CS_M36
FUNCTIONAL PROGRAMMING AND INTERACTIVE THEOREM PROVING (PART 2)
CS_M46
INTERACTIVE THEOREM PROVING
Exam May/June 2006

(*Attempt* **2** *questions out of 3*)

Question 1.

(a) (i) Let $(\mathrm{T}, \longrightarrow)$ be a reduction system. Define the equivalence relation $\longleftrightarrow^*$ between terms in T, which is derived from $\longrightarrow$.

(ii) In many reduction systems it is easy to determine for a term s the set of terms t such that $s \longrightarrow t$ holds, but difficult to determine the set of terms t such that $t \longrightarrow s$ holds. In such reduction systems it can be difficult to determine whether $s \longleftrightarrow^* t$ holds. Why?

(iii) If a reduction system as described in (ii) is confluent, there is a simpler way of determining whether $s \longleftrightarrow^* t$ holds. Which one? Justify your answer.

(iv) If a reduction systems as described in (iii) is in addition strongly normalising, it is even easier to determine, whether $s \longleftrightarrow^* t$ holds. Explain, why it indeed becomes easier.

[6 marks]

(b) Explain, in which sense the untyped λ-calculus forms a reduction system. Which of the following properties hold for this reduction system:

(i) Strong normalisation.

(ii) Weak normalisation.

(iii) Confluence.

In case a property does not hold, give a counter example.

[8 marks]

(c) Define a reduction system which is not strongly normalising and non-confluent and which uses amongst those which have these properties as few terms and reductions as possible. Show that your solution has these properties and that it is the minimal system in the following sense: there is no strongly normalising and non-confluent reduction system that has fewer terms or fewer reductions.

[5 marks]

(d) Assign types to the bound variables in the following λ-term, so that it becomes a λ-term of the simply typed λ-calculus:

$$(\lambda f.\lambda x.f\ (f\ x))\ (\lambda f.\lambda x.f\ (f\ x))$$

Determine the type of the resulting λ-term. Explain why it has this type by carrying out a derivation or determining the types of each subterm.

[6 marks]

Please turn over for Question 2

Question 2.

(a) Define depending on $A, B :: \mathrm{Set}$ the disjoint union $A + B$ of A, B in Agda.

[2 marks]

(b) The disjoint union of two sets A and B is closely related to the disjunction of two formulas A and B. Explain this relationship.

[3 marks]

(c) Define the set of Booleans. Postulate in Agda $A, B :: \mathrm{Set}$ and Boolean valued equalities $eqA :: A \to A \to \mathrm{Bool}$, $eqB :: B \to B \to \mathrm{Bool}$ on A and B, respectively. Introduce in Agda the Boolean valued equality on $A + B$ induced by eqA and eqB.

[6 marks]

(d) Introduce in Agda the function $\mathrm{atom} :: \mathrm{Bool} \to \mathrm{Set}$, which maps a truth value to the formula corresponding to this truth value. You need to introduce as well any auxiliary sets needed in this definition.

[2 marks]

(e) Introduce in Agda depending on $a, a' :: A$ the equality $EqA\ a\ a' :: \mathrm{Set}$ corresponding to $eqA\ a\ a'$. Do the same for $EqB\ b\ b'$ (depending on $b, b' :: B$) and $EqAB\ ab\ ab'$ (depending on $ab, ab' :: A + B$).

[2 marks]

(f) Introduce in Agda formulas expressing that the equalities EqA, EqB and $EqAB$ introduced in Question 2 (e) are symmetric.

[2 marks]

(g) Show the following in Agda: If EqA and EqB are symmetric, so is $EqAB$.

[8 marks]

Question 3 follows on the next page.

Question 3.

(a) How is the formula $\forall x : A.B$ represented in Agda? Explain, why one uses this representation.

[3 marks]

(b) Define the set of natural numbers N in Agda. Introduce an equality

$$(==) :: \mathrm{N} \to \mathrm{N} \to \mathrm{Set}$$

on N. You should introduce as well any auxiliary sets and functions used in your solution.

[3 marks]

(c) Introduce in Agda the relation

$$(<) :: \mathrm{N} \to \mathrm{N} \to \mathrm{Set}$$

such that $n < m$ is true if and only if n is strictly less than m.

[2 marks]

(d) State in Agda the theorem that for all natural numbers n, m, k we have that $n = m$ and $m < k$ implies $n < k$.

[3 marks]

(e) Prove the theorem stated in the previous question 3 (d).

[7 marks]

(f) A clever student claims that he can show that all natural numbers are equal to 0 (where 0 is represented as Z in Agda). His proof is as follows:

$$\begin{array}{ll} p & (n :: \mathrm{N}) \\ & ::\quad n == \mathrm{Z} \\ & =\quad p\, n \end{array}$$

What is wrong with his proof?

[3 marks]

(g) Postulate $A :: \mathrm{Set}$ in Agda. Prove the following formula in Agda:

$$((A \to A) \to A) \to A$$

[4 marks]