PRIFYSGOL CYMRU; UNIVERSITY OF WALES

DEGREE EXAMINATIONS MAY/JUNE 2002

SWANSEA

Computer Science

CS 226 Computability Theory

Attempt 2 questions out of 3

Time allowed: 2 hours

Students are permitted to use the dictionaries provided by the University

Students are NOT permitted to use calculators

CS_226
COMPUTABILITY THEORY
(*Attempt* **2** *questions out of three*)

Question 1.

(a) Define the primitive recursive functions and the partial recursive functions. Explain the main differences between them.

[9 marks]

(b) Show that the following function $f\colon \mathbf{N}^2 \to \mathbf{N}$ is primitive recursive:

$$f(x, y) := x * (x + 1) * \ldots * (x + y)$$

(for example $f(3, 0) = 3$ and $f(3, 2) = 3 * 4 * 5 = 60$).

[5 marks]

(c) Consider a function $g\colon \mathbf{N}^2 \to \mathbf{N}$ satisfying the equations

$$g(x, 0) = x$$
$$g(x, y + 1) = g(x^2, y)$$

for all $x, y \in \mathbf{N}$.

(i) Show that

$$g(x^2, y) = (g(x, y))^2 \quad \text{for all } y, x \in \mathbf{N}.$$

Hint: Use induction on y.

(ii) Use part (c) (i) to show that the function g is primitive recursive.

[7 marks]

(d) (i) What does it mean for a problem A to be reducible to a problem B?

(ii) Assume that A is an undecidable problem and that A is reducible to a problem B. What can you say about the decidability of problem B? Justify your answer.

[4 marks]

Question 2.

(a) (i) What is the Halting Problem?

(ii) Explain precisely what it means for the Halting Problem to be undecidable.

(iii) Sketch a proof of the undecidability of the Halting Problem.

[8 marks]

(b) Consider the function $\mathsf{sg}\colon \mathbf{N} \to \mathbf{N}$,

$$\mathsf{sg}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Write a URM program computing sg.

[5 marks]

(c) Let $g \colon \mathbf{N}^2 \to \mathbf{N}$ be a function with the property that for every primitive recursive function $h \colon \mathbf{N} \to \mathbf{N}$ there exists a number n such that

$$h(x) = g(n, x) \text{ for all } x \in \mathbf{N}.$$

Show that the function g is not primitive recursive. **[7 marks]**

(d) (i) How is the *Ackermann function* defined; what are its branches?

(ii) Explain why the Ackermann function and its branches are of particular interest regarding the class of primitive recursive functions and the class of computable functions.

[5 marks]

Question 3.

(a) Let M be the lambda-term $(\lambda u \lambda x(ux))(vx)$.

(i) What are the free variables of M?

(ii) Compute the normal form of M.

[6 marks]

(b) (i) Explain how natural numbers and partial functions can be represented in the lambda calculus.

(ii) What is a fixed point combinator in lambda calculus?

(iii) Fixed point combinators play crucial role in the proof that lambda definability and partial recursiveness are equivalent. In which part of this equivalence are they used?

[6 marks]

(c) (i) What is a recursively enumerable set?

(ii) What is the arithmetical hierarchy?

(iii) How does the class of recursively enumerable sets relate to the arithmetical hierarchy?

[6 marks]

(d) Let the set $A \subseteq \mathbf{N}$ be defined by $A := \{e \in \mathbf{N} \mid \{e\}^{(1)}(1) \text{ is defined}\}$.

(i) Use Turing's theorem on the existence of universal machines to prove that the set A is recursively enumerable.

(ii) Use Rice's Theorem to prove that the set A is not decidable.

(iii) Use parts (d) (i) and (ii) to show that the set $\mathbf{N} \setminus A$ is not recursively enumerable.

[7 marks]