CS_M16 Logic and Semantics **Exam**

A. Beckmann **2006**

(Attempt 2 questions out of 3)

Question 1.

(a) (i) Explain what it means for propositional logic $\mathrm{PL}[p_1, \dots, p_m]$ (with the standard connectives $\neg, \wedge, \vee$ in m propositional variables $p_1, \dots, p_m$) to be *expressively complete* for all Boolean functions of arity m. **[2 marks]**

(ii) Which of the following sets of Boolean functions form a complete base? Briefly justify your answers. (You can assume without proof that $\neg$ and $\vee$ form a complete base.)

(A) $\vee$ and $\top$ (true).

(B) $\rightarrow$ (implication) and $\perp$ (false).

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

[3 marks]

(iii) Sketch in outline one argument to prove expressive completeness of PL – with the standard connectives $\neg, \wedge, \vee$ – in m variables, for all m. **[3 marks]**

(b) Let $\varphi(p, q, r) = \big(\neg p \wedge (\neg q \vee r)\big) \ \vee \ \big(p \wedge (q \vee \neg r)\big) \ \in \ \mathrm{PL}[p, q, r]$.

Diagrammatically classify all positions in the game tree associated with the quantified Boolean formula

$$\forall p \, \exists q \, \forall r \ \varphi(p, q, r)$$

according to which of the two players, $\exists$ or $\forall$, has a winning strategy. In particular, determine whether this quantified formula evaluates to true or false. **[6 marks]**

(c) (i) Formulate the general satisfiability problem for PL, and the general model checking problem for LTL over finite traces. **[2 marks]**

(ii) How easy is satisfiability for PL; and how easy is model checking for LTL over finite traces? **[3 marks]**

(d) (i) The logics $\mathrm{LTL}[p, q]$ and $\mathrm{FOL}[<, P, Q]$ express the same properties of trace structures $\mathrm{FTS}[P, Q]$. Justify this claim. **[3 marks]**

(ii) The above claim suggests that the model checking problem for $\mathrm{FOL}[<, P, Q]$ over finite traces is as easy as the model checking problem for $\mathrm{LTL}[p, q]$ over finite traces. What is wrong with this argument? **[3 marks]**

Question 2.

Consider FOL$[<, P, Q]$ and LTL$[p, q]$ over finite trace structures.

(a) State in words the temporal properties expressed by the following LTL formulae φ; in each case also sketch an example of a trace structure $\mathcal{A} \in \mathrm{FTS}[P, Q]$ with designated t that satisfies φ, and one that does not satisfy φ.
For both formulae φ provide natural translations $\hat{\varphi}(x) \in \mathrm{FOL}$ such that $\hat{\varphi}$ is logically equivalent to φ over $\mathrm{FTS}[P, Q]$.

(i) $\varphi = \mathbb{G}\big(p \vee \mathbb{X}p\big)$.

(ii) $\varphi = \big(\mathbb{F}p\big)\, \mathbb{U}\, q$.

[8 marks]

(b) Express the following trace properties in LTL:

(i) At some point from now on, p will become true at two consecutive time steps.

(ii) p is true now or will never become true from now on.

[6 marks]

(c) Systematically evaluate the LTL formula $\varphi = \neg\mathbb{G}\big(p\mathbb{U}(\neg q)\big)$ and all its relevant subformulae over the trace structure

$$\mathcal{A} = \big(\{1, 2, 3, 4\}, P^{\mathcal{A}}, Q^{\mathcal{A}}\big) \quad \text{where} \quad P^{\mathcal{A}} = \{1, 3\}, \text{ and } Q^{\mathcal{A}} = \{2, 3\},$$

following the dynamic programming idea for evaluation in backward time direction. Determine whether $\mathcal{A} \models \varphi$ holds true or not. **[6 marks]**

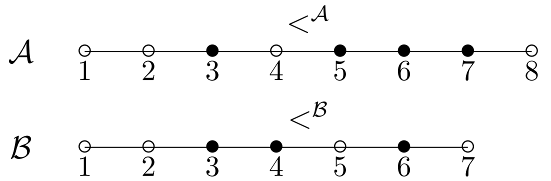
(d) (i) Outline in words an automata theoretic method for checking the formula φ from part (c) for satisfiability over $\mathrm{FTS}[P, Q]$.
In particular describe the nature and intended meaning of the states and of the transitions of the relevant automaton $\mathcal{M}_\varphi$. Illustrate two sample transitions in $\mathcal{M}_\varphi$, one from the initial state and one other.

(ii) Which property of $\mathcal{M}_\varphi$ is related to satisfiability of φ, and how can it be checked?

[5 marks]

Question 3

(a) Consider the FOL Ehrenfeucht-Fraïssé game over the two finite trace structures $\mathcal{A}, \mathcal{B}$ depicted in the diagram below.



$P^{\mathcal{A}} = \{3, 5, 6, 7\}$

$P^{\mathcal{B}} = \{3, 4, 6\}$

(i) Give a configuration with one pebble in each structure from which Player **I** can win with just one more round; and one with five pebbles in each structure (all on different points) from which Player **II** still has a winning strategy for one more round. **[3 marks]**

(ii) Show that $\mathcal{A} \equiv_2 \mathcal{B}$ by describing how Player **II** should respond to all possible first moves of Player **I** so as to be able to respond to her second move. **[3 marks]**

(iii) Give a FOL formula of quantifier rank 3 that distinguishes between $\mathcal{A}$ and $\mathcal{B}$, and also an LTL formula that distinguishes $\mathcal{A}, 1$ from $\mathcal{B}, 1$. **[3 marks]**

(b) (i) State and explain the FOL Ehrenfeucht-Fraïssé Theorem in its form for finite trace structures $\mathcal{A}, \mathcal{B} \in \mathrm{FTS}[]$, that is, finite linear orders $\mathcal{A} = (A, <^{\mathcal{A}})$ and $\mathcal{B} = (B, <^{\mathcal{B}})$. **[4 marks]**

(ii) The Ehrenfeucht-Fraïssé Theorem can be used to give a characterisation of the level of indistinguishability $\equiv_m$ between two finite linear orders in terms of their lengths. Formulate this characterisation. Explain how it can be used to show that the property $\mathcal{P}_{\mathrm{odd}}$ of finite linear orders is not FOL[<] definable, where $\mathcal{P}_{\mathrm{odd}}$ holds true for a finite linear order $\mathcal{A}$ iff $\mathcal{A}$ is of odd length. **[3 marks]**

(iii) Define a formula φ_{odd} in MSOL[<] which defines, over the class of finite linear orders, exactly those of odd length. **[3 marks]**

(iv) Are the logics LTL[] and MSOL[<] equal in expressive power? Explain your answer. **[2 marks]**

(c) Carefully state and explain the meaning of Büchi's Theorem.

Illustrate with an example how it can be used to show that certain properties of finite trace structures are not definable in monadic second-order logic MSOL. **[4 marks]**