CS 132
Algorithms and Computation
(Answer 2 questions out of 3)

Question 1

Consider the following algorithm, which takes an array $A[1..n]$ of integers and its length $n \geq 1$ as its inputs:

SPHINX (A, n)

```
m ← A[1]
i ← 2
while i ≤ n do
        if m > A[i] then m ← A[i]
        i ← i + 1
output m
```

a) Execute the algorithm SPHINX for the following inputs:

i) $A = [17], n = 1$

ii) $A = [1, -4], n = 2$

iii) $A = [4, 1, 42], n = 3$

Use for this the following scheme:

- $A = \ldots, \; n = \ldots$
-

PC	m	i	comments
...	...	...	...

- The output is ...

[6 marks]

b) Formulate the computational problem which is solved by SPHINX.

[4 marks]

c) Argue carefully why SPHINX terminates for all inputs.

[5 marks]

(Question 1 continues on page 2.)

d) Consider the running time of SPHINX, where comparisons have cost 1, and all other operations are free (i.e. the execution of the lines 1, 2, 5 & 6 does not incur any costs).

i) Give the total running time for the following inputs:

- $A = [17]$,
- $A = [1, -4]$,
- $A = [4, 1, 42]$.

[3 marks]

ii) Give a formula for the running time depending on the number n of elements in the array A. Give also a tight bound for the running time using Θ.

[4 marks]

e) Prove: If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

[3 marks]

Question 2

a) Prove the following:

   i) $3n^2 - 2n + 4 = O(n^2)$
   ii) $3n^2 - 4n + 12 = \Omega(n^2)$

[6 marks]

b) Consider the following algorithm, which takes two natural numbers a and b as input, and returns their product $a * b$ as its output using as operations addition and subtraction of natural numbers only.

MULT (a, b)

$d \leftarrow 0$
$i \leftarrow a$
while $i > 0$ **do**
  $d \leftarrow d + b$
  $i \leftarrow i - 1$
output d

Formulate a recursive variant MULT $(a, b; d, i)$ of this algorithm, where a and b are the numbers to be multiplied, d stores the intermediate results, and i is used as a counter.

The call MULT $(a, b; 0, a)$ shall produce $a * b$ as its output.

[7 marks]

c) Give regular expressions that represent the following sets:

   i) The set of strings over the alphabet $\{a, b, c\}$ that have length three.
   ii) The set of strings over the alphabet $\{a, b, c\}$ that contain the substrings aa and bb.
   iii) The set of strings over the alphabet $\{a, b, c\}$ that do not begin with aaa.

[12 marks]

Question 3

a) Let v and w be strings over an alphabet Σ. Give the definitions for:

i) v is a substring of w.

ii) v is a prefix of w.

iii) v is a suffix of w.

[6 marks]

b) Let $\Sigma = \{a, b, c\}$ be an alphabet. Prove each of the following:

i) ab is a substring of $abbc$.

ii) bc is a suffix of $abbc$.

[4 marks]

c) Let

$$L = \{w \in \{a, b, c\}^* \mid \text{every } b \text{ is immediately followed by at least one } c\}$$

be a language over the alphabet $\Sigma = \{a, b, c\}$.

i) Give an example of a word which belongs to L, and an example of a word which does not belong to L.

ii) Draw the graph of a deterministic, finite automaton which recognises L.

[8 marks]

d) Let L be the set of strings over $\{a, b, c\}$ that contain either aa or bb (or both) as substrings.

i) Give an example of a word which belongs to L, and an example of a word which does not belong to L.

ii) Draw the graph of a non-deterministic, finite automaton which recognises L.

[7 marks]