CS 132
Algorithms and Computation
(Answer 2 questions out of 3)

Question 1

a) Show the execution steps of the following imperative algorithm for the input $X = 8,\ Y = 12$, and give the result of its computation.

```
GCD:
var       X, Y : int;
input     X; Y
      1   while X ≠ Y   do
      2     if X > Y
      3                 then   X:= X-Y
      4                 else   Y:= Y-X
      5     fi
      6   od
output    X
```

For this purpose use a table of the form

Program Counter	X	Y
0	8	12
1	...	...
...	...	...

which shows the value of the variables *after* execution of the command in the line indicated by the Program Counter.

[5 marks]

b) Write three imperative algorithms:

i) one algorithm which never terminates.

ii) one algorithm which tests if a given integer ≥ 2 is prime; its input should be of type int, and its output of type bool.

iii) one algorithm which returns the next prime number greater than or equal to its input; its input should be of type int, and its output of type int.

[14 marks]

c) Prove the following:

i) Let $f(n) = 3n + 2$. Prove that $f(n) = O(n)$.

ii) Let $g(n) = 4n^2 + 5n + 17$. Prove that $g(n) = O(n^2)$.

[6 marks]

Question 2

a) Evaluate the function call

$$isSorted(cons(4, cons(2, nil))),$$

where $isSorted$ is the following applicative algorithm:

$$\begin{array}{lcl} isSorted(l : Tintlist) & = & \underline{if}\ length(l) = 0 \vee length(l) = 1 \\ & & \underline{then}\ true \\ & & \underline{else}\ hd(l) \leq hd(tail(l)) \wedge isSorted(tail(l))\ \underline{fi} \\ length(l : Tintlist) & = & \underline{if}\ l = nil\ \underline{then}\ 0\ \underline{else}\ 1 + length(tail(l))\ \underline{fi} \end{array}$$

You are allowed to leave out some intermediate evaluation steps; it is enough to show about 4 or 5 evaluations leading to the final result.

[5 marks]

b) Recall the definition of int terms:

I1 The integers $\ldots, -2, -1, 0, 1, 2, \ldots$ as well as integer variables $x, y, \ldots$ are int-terms.

I2
- $(t + u)$,
- $(t * u)$,
- $(t - u)$,
- $(t\ div\ u)$,
- $(t\ mod\ u)$,
- $abs(t)$,
- $sign(t)$,
- if b then t else u fi

are int terms if t and u are int terms and b is a bool term.

I3 Only those strings defined by **I1** and **I2** are int terms, no others.

Show that

i) '$(2 + (3 * x))$' and '$((y - 4)\ mod\ 2)$' are int terms;

ii) '$\underline{if}\ (2 > 4)\ \underline{then}\ 3\ \underline{else}\ false\ \underline{fi}$' is *not* an int term.

[5 marks]

c) Write two applicative algorithms:

i) One algorithm where the evaluation of a function call never terminates.

ii) One algorithm *DeleteDoubles* which takes a list of integers and removes all doubles, preserving the order of the list elements; for example:

$$cons(1, cons(2, cons(1, cons(3, cons(1, nil)))))$$

becomes

$$cons(1, cons(2, cons(3, nil))).$$

You can use all applicative algorithms introduced in the course like

length, append, front, last

without repeating their definition. *But* you need to define any new function which you are using.

[10 marks]

d) Prove by induction: for all $l \in \underline{intlist}$,

$$length(cons(42, l)) \geq 1$$

where

$$length(l : \underline{intlist}) = \underline{if}\ l = nil\ \underline{then}\ 0\ \underline{else}\ 1 + length(tail(l))\ \underline{fi}.$$

[5 marks]

Question 3

a) Let Σ be an alphabet. Give the definition of the sets

i) Σ^k, $k \geq 0$,

ii) Σ^+, and

iii) Σ^*.

[6 marks]

b) Let $L := \{w \in \{0,1\}^* \mid w \text{ has an even number of } 0's \text{ and an odd number of } 1's\}$ be a language over the alphabet $\Sigma = \{0,1\}$.

i) Give an example of a word which belongs to L, and an example of a word which does not belong to L.

ii) Draw the graph of a deterministic, finite automaton which recognises L.

[10 marks]

c) Consider the data type nat of natural numbers.

i) Define its values.

ii) For each of the operations $+, *, -, div, mod, \leq$ on natural numbers

- give the profile of the operation and
- present two illustrating examples.

[9 marks]