CS_316 (2004-05)
Logic and Semantics
(*Attempt 2 questions out of 3*)

Question 1

(a) (i) Given an assignment $\mathbf{a} \in \{0,1\}^m$ to m propositional variables $p_1,\ldots,p_m$, define its ***characteristic formula***, that is, the formula $\chi_{\mathbf{a}}(p_1,\ldots,p_m) \in \mathrm{PL}[p_1,\ldots,p_m]$ such that, for all $\mathbf{b} \in \{0,1\}^m$:

$$\mathbf{b} \models \chi_{\mathbf{a}}(p_1,\ldots,p_m) \quad \text{iff} \quad \mathbf{b} = \mathbf{a}.$$

[2 marks]

(ii) What does it mean for a PL formula $\varphi(p_1,\ldots,p_m)$ to be in ***disjunctive normal form*** (DNF)?

Using characteristic formulae $\chi_{\mathbf{a}}$, show how you can transform any PL formula $\varphi(p_1,\ldots,p_m) \in \mathrm{PL}[p_1,\ldots,p_m]$ into DNF. **[3 marks]**

(iii) Determining if a PL formula in DNF is ***satisfiable*** is easy: it can be done in time which is linear in the size of the formula. Explain how this can be done? **[2 marks]**

(iv) It would appear then that determining the satisfiability of any given PL formula is easy: we merely need to transform the formula into DNF and test if this equivalent formula is satisfiable.

What is wrong with this argument? How easy is the satisfiability problem for PL in reality? **[3 marks]**

(b) Let $\varphi(p,q,r) = \big(q \wedge \neg r\big) \ \vee \ \big(\neg p \wedge (q \vee r)\big) \ \in \ \mathrm{PL}[p,q,r]$.

Diagrammatically classify all positions in the game tree associated with the quantified Boolean formula

$$\exists p\, \forall q\, \exists r\ \varphi(p,q,r)$$

according to which of the two players, $\exists$ or $\forall$, has a winning strategy. In particular, determine whether this quantified formula evaluates to true or false. **[6 marks]**

(Please turn over.)

(c) Let $\varphi(p, q, r)$ be as in part (b). For the analysis of games associated with its quantified versions $Q^{(1)}p\, Q^{(2)}q\, Q^{(3)}r\, \varphi$ consider the transition system

$$\mathcal{A}_\varphi = (A, W^{\mathcal{A}_\varphi}, E^{\mathcal{A}_\varphi}) \quad \text{where} \quad \begin{array}{ll} A & = \{\varepsilon\} \cup \{0,1\} \cup \{0,1\}^2 \cup \{0,1\}^3, \\ W^{\mathcal{A}_\varphi} & = \{\mathbf{a} \in \{0,1\}^3 \colon \mathbf{a} \models \varphi(p,q,r)\}, \\ E^{\mathcal{A}_\varphi} & = \{(\mathbf{a}, \mathbf{a}') \in A \times A \colon \mathbf{a}' = \mathbf{a}0 \text{ or } \mathbf{a}' = \mathbf{a}1\}. \end{array}$$

Its domain is the set of game positions (partial assignments) with initial position ε (the empty assignment); W (or a new basic proposition w for "win") marks those final positions in which player $\exists$ wins; E encodes all possible moves (for either player).

(i) Sketch $\mathcal{A}_\varphi$. **[1 marks]**

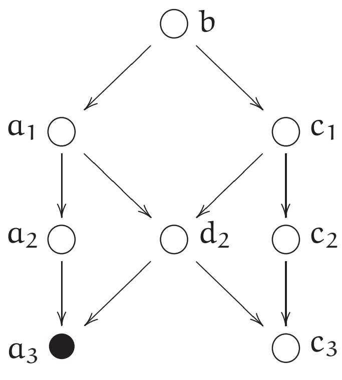
(ii) Give a natural translation $\varphi_{\exists\forall\exists} \in \mathrm{PML}[w; E]$ of the formula $\exists p\, \forall q\, \exists r\, \varphi$ from part (b), that captures the existence of a strategy for player $\exists$ as a property of the initial state ε in $\mathcal{A}_\varphi$. Check whether $\mathcal{A}_\varphi, \varepsilon \models \varphi_{\exists\forall\exists}$ by systematically evaluating the relevant sub-formulae in $\mathcal{A}_\varphi$. **[2 marks]**

(iii) Indicate the general translation of quantified versions $Q^{(1)}p\, Q^{(2)}q\, Q^{(3)}r\, \varphi$, where $Q^{(i)} \in \{\forall, \exists\}$, into $\varphi_{Q^{(1)}Q^{(2)}Q^{(3)}} \in \mathrm{PML}[w; E]$, such that

$$\mathcal{A}_\varphi, \varepsilon \models \varphi_{Q^{(1)}Q^{(2)}Q^{(3)}} \quad \text{iff} \quad Q^{(1)}p\, Q^{(2)}q\, Q^{(3)}r\, \varphi \ \text{ evaluates to true.}$$

[2 marks]

(d) Consider the following transition system $\mathcal{B} \in \mathrm{FS}[W; E]$, in which the only element of $W^{\mathcal{B}}$ is a_3, marked $\bullet$.



(i) Indicate a bisimulation between $\mathcal{B}, b$ and $\mathcal{A}_\varphi, \varepsilon$ (the transition system from part (c)), to show that $\mathcal{B}, b \sim \mathcal{A}_\varphi, \varepsilon$. **[2 marks]**

(ii) Use $\mathcal{B}, b$ and the modal formulae $\varphi_{Q^{(1)}Q^{(2)}Q^{(3)}}$ from Part (c)(iii) to find two other fully quantified versions of φ, one that evaluates to true and one that evaluates to false. **[2 marks]**

Question 2

The syntax of $\mathrm{LTL}[p, q]$ is given as follows:

- $\top$, $\bot$, p and q are (atomic) formulae; and
- if φ and ψ are formulae then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\mathbb{X}\varphi$, $\mathbb{F}\varphi$, $\mathbb{G}\varphi$ and $\varphi\mathbb{U}\psi$.

(a) Give the complete inductive definition of the satisfaction relation $\mathcal{A}, t \models \varphi$, where $\varphi \in \mathrm{LTL}[p, q]$ and $\mathcal{A} \in \mathrm{FTS}[P, Q]$ is a finite trace structure over the usual domain $A = \{1, 2, \ldots, \ell\}$ with a distinguished point $t \in A$. **[6 marks]**

(b) Two of the following equivalences are correct, and the other two are false:

$$\mathbb{F}(\varphi \wedge \psi) \equiv \mathbb{F}\varphi \wedge \mathbb{F}\psi \qquad \mathbb{F}(\varphi \vee \psi) \equiv \mathbb{F}\varphi \vee \mathbb{F}\psi$$
$$\mathbb{G}(\varphi \wedge \psi) \equiv \mathbb{G}\varphi \wedge \mathbb{G}\psi \qquad \mathbb{G}(\varphi \vee \psi) \equiv \mathbb{G}\varphi \vee \mathbb{G}\psi$$

(i) Which two are correct? Prove both of these. **[3 marks]**

(ii) Prove that the other two are incorrect. **[3 marks]**

(c) State precisely in which sense every $\mathrm{LTL}[p, q]$ formula can be translated into an equivalent $\mathrm{FOL}[<, P, Q]$ formula, and give a complete definition of this translation. **[5 marks]**

(d) Suppose we are modelling a *resource*, and p represents a *request* for the resource while q represents the *granting* of the resource in response to such a request. We want the following properties to be true. (They are each expressed twice, for clarity.)

(i) The resource cannot already be requested or granted at the start:

Initially both p *and* q *are false.*

(ii) The resource cannot be granted unless it has been requested; and when it is granted, the request will be satisfied and thus withdrawn:

q *can never become true unless* p *is true at the previous moment; and when* q *becomes true,* p *will become false.*

(iii) A request for the resource will remain in place until it is granted, which must eventually happen, but not while the resource is granted to a previous request:

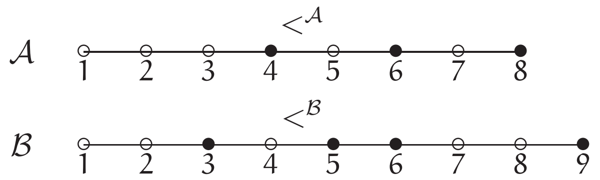
Whenever p *is true, it will remain true until* q *becomes true (that is, until* q *changes from false to true).*

Express each of these properties in both $\mathrm{LTL}[p, q]$ and in $\mathrm{FOL}[<, P, Q]$. **[8 marks]**

Question 3

(a) State and explain the FOL Ehrenfeucht-Fraïssé Theorem in its form for finite trace structures $\mathcal{A}, \mathcal{B} \in \mathrm{FTS}[P]$. **[4 marks]**

(b) Consider the FOL Ehrenfeucht-Fraïssé game over the two finite trace structures $\mathcal{A}, \mathcal{B}$ depicted in the diagram below.



$P^{\mathcal{A}} = \{4, 6, 8\}$

$P^{\mathcal{B}} = \{3, 5, 6, 9\}$

(i) Give a configuration with one pebble in each structure from which Player **I** can win with just one more round; and one with four pebbles in each structure (all on different points) from which Player **I** cannot win with just one more round. **[3 marks]**

(ii) Show that $\mathcal{A} \equiv_2 \mathcal{B}$ by describing how Player **II** should respond to all possible first moves of Player **I** so as to be able to respond to her second move. **[3 marks]**

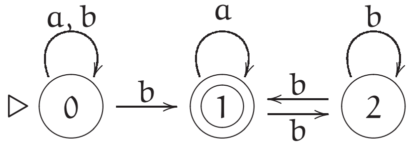
(iii) Give a FOL formula of quantifier rank 3 that distinguishes between $\mathcal{A}$ and $\mathcal{B}$, and also an LTL formula that distinguishes $\mathcal{A}, 1$ from $\mathcal{B}, 1$. **[3 marks]**

(c) Can there be finite trace structures $\mathcal{A}, \mathcal{B} \in \mathrm{FTS}[P]$ that are not isomorphic, yet for which Player **II** has a winning strategy for any number of rounds on $\mathcal{A} \mid \mathcal{B}$? Explain. **[3 marks]**

(d) Carefully state and explain the meaning of Büchi's Theorem.

Illustrate with an example how it can be used to show that certain properties of finite trace structures are not definable in monadic second-order logic MSOL. **[4 marks]**

(e) Give a sentence in $\mathrm{MSOL}[P_a, P_b]$ that defines the class of those word structures $\mathcal{A}_w \in \mathrm{WS}[\{a, b\}]$ associated with the regular language accepted by the following finite state automaton, as in the proof of Büchi's Theorem.



Briefly explain the structure of your sentence. You may assume as given the usual auxiliary first-order formulae $\chi_{\text{next}}(x, y)$, $\chi_{\text{first}}(x)$, $\chi_{\text{last}}(x)$. **[5 marks]**