PRIFYSGOL CYMRU; UNIVERSITY OF WALES

DEGREE EXAMINATIONS MAY/JUNE 2003

SWANSEA

Computer Science

CS 342 Constraint Satisfaction Problems and Applications

Attempt 2 questions out of 3

Time allowed: 2 hours

Students are permitted to use the dictionaries provided by the University

Students are NOT permitted to use calculators

CS_342 2002/03
Constraint Satisfaction Problems and Applications

(Attempt 2 questions out of 3)

Question 1 Clause-sets and partial assignments

(a) Basic notions

(i) Define variables ($\mathcal{VA}$), literals ($\mathcal{LIT}$), clauses ($\mathcal{CL}$) and clause-sets ($\mathcal{CLS}$), and explain the meaning of these concepts.

(ii) Define partial assignments ($\mathcal{PASS}$), and explain the operation $* : \mathcal{PASS} \times \mathcal{CLS} \to \mathcal{CLS}$ of partial assignments on clause-sets; that is, explain how $\varphi * F$ is computed from F.

(iii) Compute $\langle a \to 0, b \to 1 \rangle * \{ \{\overline{a}, \overline{b}, c\}, \{a, b, \overline{c}\}, \{a, d\}, \{\overline{c}, \overline{d}\} \}$.

(iv) Explain the composition $\varphi \circ \psi$ of partial assignments.

(v) Compute $\langle a \to 0, b \to 1, c \to 1 \rangle \circ \langle a \to 0, b \to 0, d \to 1 \rangle$.

(vi) State the basic properties of the composition of partial assignments and the operation of partial assignments on clause-sets.

(vii) Given a clause C, what is the corresponding partial assignment φ_C, and given a partial assignment φ, what is the corresponding clause C_φ ?

[12 marks]

(b) Satisfiability and unsatisfiability

(i) Define the set $\mathcal{SAT}$ of satisfiable clause-sets and the set $\mathcal{USAT}$ of unsatisfiable clause-sets.

(ii) Let $F := \{ \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_4\}, \{\overline{v_4}, v_1\}, \{v_1, v_2, v_3, v_4\}, \{\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}\} \}$. Decide whether F is satisfiable or not, and prove your answer.

(iii) Prove that if a clause-set F is satisfiable, then also every sub-clause-set $F' \subseteq F$ is satisfiable.

[6 marks]

(c) Full clause-sets

(i) Define the notion of a "full clause-set".

(ii) Give a (good) upper bound on $c(F)$ for full clause-sets F in terms of $n(F)$.

(iii) Describe a fast decision procedure for the satisfiability problem on full clause-sets, and prove the correctness of your algorithm.

[7 marks]

Question 2 Autarkies and 2-CNF

(a) Autarkies and their basic properties

(i) Explain in words what an *autarky* is.

(ii) Define formally $\mathrm{Auk}(F)$, the set of autarkies for a clause-set F, using a different formulation for autarkies than in part (i).

(iii) State the three main properties of autarkies (their relation to satisfying assignments, their "harmlessness" and about composition), and give reasons, why these properties hold true (if possible, prove your assertions).

(iv) Call an autarky φ for F *non-trivial* iff φ satisfies at least one clause from F. How is it possible to decide whether F has a non-trivial autarky?! Develop some algorithmic ideas (try to relate some time bounds to these ideas).

[13 marks]

(b) Binary clause-sets

(i) How is $2\text{–}\mathcal{CLS}$ defined?

(ii) Consider $F \in 2\text{–}\mathcal{CLS}$. Show that all clauses in $r_1(F)$ have the same length. (Hint: Make a case distinction, whether a contradiction was found or not.)

(iii) Show that in case of $r_1(\langle v \to \varepsilon \rangle * F) \neq \{\bot\}$ for some variable v and $\varepsilon \in \{0, 1\}$ the clause-set $r_1(\langle v \to \varepsilon \rangle * F)$ is satisfiability equivalent to $F \in 2\text{–}\mathcal{CLS}$.

(iv) Give an algorithm in pseudo-code for deciding satisfiability of clause-set $F \in 2\text{–}\mathcal{CLS}$ in polynomial time. Explain how it works, and why the running time is polynomially bounded.

[12 marks]

Question 3 Unit Propagation and Horn Formulas

(a) Unit-clause propagation

(i) State the definition of $r_1 : \mathcal{CLS} \to \mathcal{CLS}$, and also explain the definition in words.

(ii) Explain why the computation of $r_1(F)$ always terminates.

(iii) Explain why $r_1(F)$ is independent of the choices made in the course of the computation.

(iv) Sketch a proof of the fact that $r_1(F)$ and F are satisfiability equivalent.

(v) Sketch a proof of the fact that for every clause-set F there exists a partial assignment φ with $r_1(F) = \varphi * F$.

[11 marks]

(b) Fast implementation

(i) State the run time and space complexity of the naive implementation of unit propagation.

(ii) What are the main ideas for implementing a linear time algorithm for unit propagation?

(iii) The linear time algorithm for unit propagation consists of a double nested loop (running through all unit clauses found, and applying them to the formula). How is it that the running time in fact is linear and not quadratic?

[8 marks]

(c) Horn clause-sets

(i) Define a Horn clause-set.

(ii) Prove that if a Horn clause-set F does not contain the empty clause or a unit clause, then F must be satisfiable.

(iii) Show how to decide satisfiability of Horn clause-sets in linear time.

[6 marks]