CS_116
MODELLING COMPUTING SYSTEMS
(*Attempt 2 questions out of 3*)

Question 1

(a) In the game of **NIM**, an arbitrary number of piles of coins are formed, each with an arbitrary number of coins in them, and two players alternate in removing one or more coins from any one pile. Whoever takes the last coin is declared to be the winner.

Assuming both players play optimally, determine which player will win the game of NIM when played with three piles containing the following numbers of coins:

(i) 2, 5, 6

(ii) 3, 5, 6

(iii) 4, 5, 6

(iv) 4, 5, 7

In each case, justify your answer referring to the general theory of NIM (balanced versus unbalanced positions). In the cases in which the first player will win, give *all* possible winning first moves.

[10 marks]

(b) If a player is in a winning position in NIM, then there will in general be more than one winning move. (Two moves are different if they involve different piles, or if they involve the same pile but removing different numbers of coins.) What is the maximum number of different winning moves possible from a NIM position with n piles? Justify your answer.

[3 marks]

(c) In **1-4 NIM**, there is only one pile, and the two players alternately remove either 1 or 4 coins from the pile. Again, the player to take the last coin wins.

(i) For each number n from 1 to 10, explain who has the winning strategy in 1-4 NIM starting from a pile of n coins. In the cases in which the first player has the winning strategy, state how many coins (1, 4, or either) the first player should take.

[5 marks]

(ii) Do the same for **MISÈRE 1-4 NIM**, in which the player to take the last coin loses.

[4 marks]

(iii) If you are the first player in a game of **1-4 NIM** starting with 100 coins, would you like to play the regular game or the **Misère** version? Why?

[3 marks]

Question 2

(a) Let

$$A \stackrel{\text{def}}{=} b.c.\mathbf{0} + b.d.\mathbf{0} \qquad C \stackrel{\text{def}}{=} a.B + a.A$$
$$B \stackrel{\text{def}}{=} A + b.(c.\mathbf{0} + d.\mathbf{0}) \qquad D \stackrel{\text{def}}{=} a.B$$

(i) Draw a process graph which includes the above states A, B, C and D.

[5 marks]

(ii) Give a formula P of the modal logic M which is satisfied by C but not by D.

[5 marks]

(iii) Argue why (or why not) the formula you gave in (ii) is of minimal modal depth (that is, that no formula of smaller modal depth could distinguish between C and D).

[5 marks]

(b) Design a simple change-making process which will initially accept a 5p, 10p or 20p coin, and dispense any sequence of 1p, 2p and 5p coins which sum up to the value of the coin inserted before returning to its initial state.

To do this, introduce the following events:

$i5$: insert a 5p coin	$d1$: dispense a 1p coin
$i10$: insert a 10p coin	$d2$: dispense a 2p coin
$i20$: insert a 20p coin	$d5$: dispense a 5p coin

as well as the process variables C_n for $n \in \{0, 1, 2, \ldots, 20\}$.

The process variable C_0 is to represent the initial state of the process, and has the following definition:

$$C_0 \stackrel{\text{def}}{=} i5.C_5 + i10.C_{10} + i20.C_{20}$$

(i) Give the definitions for the remaining process variables $C_1, C_2, \ldots, C_{20}$.

[4 marks]

(ii) Give a smallest (in terms of modal depth) formula of the modal logic M which is satisfied by C_5 but not by C_4.

[3 marks]

(iii) Give a smallest (in terms of modal depth) formula of the modal logic M which is satisfied by C_5 but not by C_6.

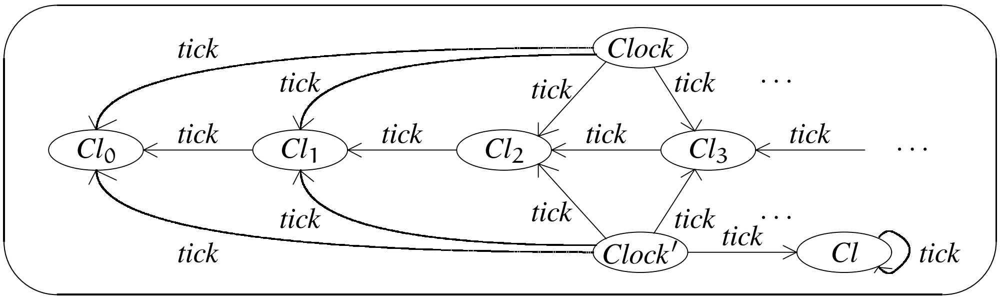
[3 marks]

Question 3

(a) Recall the following clock processes.

$$Cl \stackrel{\text{def}}{=} tick.Cl \qquad Cl_0 \stackrel{\text{def}}{=} \mathbf{0} \qquad Clock \stackrel{\text{def}}{=} \sum_{i \geq 0} Cl_i$$

$$Cl_{n+1} \stackrel{\text{def}}{=} tick.Cl_n \qquad Clock' \stackrel{\text{def}}{=} Clock + Cl$$



(i) Explain why $Clock \sim_n Clock'$ for every $n \in \mathbb{N}$. That is, explain how the second player (Bob) can win any bisimulation game of any length $n \in \mathbb{N}$ which starts with the tokens on *Clock* and *Clock*′.

[4 marks]

(ii) Explain why $Clock \not\sim Clock'$.

[4 marks]

(iii) Explain why no formula of the modal logic M can distinguish between *Clock* and *Clock*′.

[4 marks]

(iv) Prove, by induction on n, that for all $n \in \mathbb{N}$ $Cl_n \sim_n Cl_{n+1}$.

[4 marks]

(b) Consider the following two statements about a computer:

(i) "The computer consists of three parts: a CPU, a memory unit, and a bus for communication with the environment."

(ii) "The emergency button can be pushed; this will halt the computer, which will then not do anything further."

One of these statements can be expressed in the modal logic M; express it.

[3 marks]

The other statement cannot be formalized in M; explain why not.

[3 marks]

(c) Give an example of a pair of processes E and F and a pair of formulas P and Q of the modal logic M such that

$$E \models P \quad \text{and} \quad F \models Q \quad \text{but} \quad E + F \not\models P \wedge Q.$$

[3 marks]