PRIFYSGOL CYMRU; UNIVERSITY OF WALES

DEGREE EXAMINATIONS JANUARY 2002

SWANSEA

Computer Science

CS 376 Programming with Abstract Data Types

Attempt 2 questions out of 3

Time allowed: 2 hours

Students are permitted to use the dictionaries provided by the University through the invigilators

CS_376
PROGRAMMING WITH ABSTRACT DATA TYPES
(Attempt 2 questions out of 3)

Question 1.

(a) Define the notions of a

homomorphism,
epimorphism,
monomorphism,
isomorphism

between two Σ-algebras.

(5 marks)

(b) (i) What is a *congruence* on a Σ-algebra?

(ii) Let $\Sigma = (S, \Omega)$ be a signature and let $\varphi\colon A \to B$ be a homomorphism between Σ-algebras A and B. Let $\sim$ be a congruence on B. For every sort $s \in S$ define a binary relation $\approx_s$ on A_s by

$$a \approx_s a' \qquad :\Longleftrightarrow \qquad \varphi_s(a) \sim_s \varphi_s(a')$$

Show that

$$\approx \;:=\; (\approx_s)_{s\in S}$$

is a congruence on A.

(10 marks)

(c) Let Σ be the signature consisting of one sort s, a constant $0\colon s$ and one binary operation $+\colon s \times s \to s$.

Consider the Σ-algebras A, B, C with the following carrier sets:

$$A_s := \{0, 1, 2, 3, \ldots\}$$
$$B_s := \{0, 2, 4, 6, \ldots\}$$
$$C_s := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

In each of the algebras A, B, C the constant 0 is to be interpreted by the number 0, and the operation $+$ is to be interpreted by ordinary addition restricted to the corresponding carrier set.

For each of the algebras B and C decide whether or not it is isomorphic to the algebra A. Justify your answers.

(6 marks)

(d) What is an abstract data type? When is an abstract data type called monomorphic?

(4 marks)

Question 2.

(a) Produce an initial specification of the algebra of *finite lists of boolean values* that contains (among others) operations testing whether or not

- all members of a given list are equal to the boolean value #t,
- #t occurs in a given list,
- a given list is an initial segment of another.

(9 marks)

(b) What is *Rapid Prototyping* for an initial specification? Explain its use and describe under which conditions it can be applied.

(6 marks)

(c) State *Birkhoff's Theorem* on the completeness of equational logic. Explain the notions involved.

(6 marks)

(d) Consider the term rewriting system given by the rules

$$g(x, x) \mapsto c$$
$$g(f(x), f(y)) \mapsto f(g(x, y))$$

where $c\colon s$ is a constant, $f\colon s \to s$ and $g\colon s \times s \to s$ are operations, and $x, y\colon s$ are variables.

Show that this term rewriting system is terminating, but not confluent.

(4 marks)

Question 3.

(a) (i) Define what it means for a Σ-algebra A to be *initial in a class* $\mathcal{C}$ of Σ-algebras.

(ii) Explain the difference between a *loose specification* and an *initial specification*, and sketch how a model for an initial specification can be constructed.

(7 marks)

(b) Let NAT be the loose specification

Loose Spec	
Sorts	nat
Constants	zero: nat
Operations	succ: nat $\to$ nat
	pred: nat $\to$ nat
Variables	x: nat
Axioms	$\mathrm{pred}(\mathrm{succ}(x)) = x$

Let **ZSP** be the signature of NAT.

(i) Show that the following **ZSP**-algebra N is a model of NAT:

$$\mathrm{N}_{\mathrm{nat}} := \{0, 1, 2, 3, \ldots\}$$
$$\mathrm{zero}^{\mathrm{N}} := 0$$
$$\mathrm{succ}^{\mathrm{N}}(n) := n + 1$$
$$\mathrm{pred}^{\mathrm{N}}(n) := \begin{cases} n - 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

(ii) Show that N is *not* initial in the class of all models of NAT.

(6 marks)

(c) Let INT be the initial specification that is obtained from the loose specification NAT above by replacing the keywords **Loose Spec** and **Axioms** by **Init Spec** and **Equations** respectively, and adding the equation

$$\mathrm{succ}(\mathrm{pred}(x)) = x$$

(i) Describe the closed **ZSP**-terms that are in normal form with respect to the term rewriting system associated with INT.

(ii) Show that the following **ZSP**-algebra Z is a model of INT:

$$\mathrm{Z}_{\mathrm{nat}} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$
$$\mathrm{zero}^{\mathrm{Z}} := 0$$
$$\mathrm{succ}^{\mathrm{Z}}(n) := n + 1, \qquad \mathrm{pred}^{\mathrm{Z}}(n) := n - 1$$

(iii) Are there models of INT that are not isomorphic to Z? Justify your answer.

(12 marks)