Theory of Programming Languages
January 2006

Question 1

(a) Define the *Chomsky Hierarchy*. Define the *recognition problem* for the language *L(G)* defined by a grammar *G* and discuss the algorithmic properties of the recognition problem for each of the four grammars of the *Chomsky Hierarchy*.

[9 marks]

(b) Give a context-free grammar that defines a language of signatures. Illustrate the use of your grammar by *sketching* the basic steps in the derivation of some signature from your grammar, such as:

signature	Booleans
sorts	bool
constants	true: → bool
operations	and: bool × bool → bool;
	not: bool → bool
end	

Does your grammar define all the desirable conditions on the language of signatures and if not why not?

[6 marks]

(c) State carefully the *Pumping Lemma* for context-free languages.

[3 marks]

(d) State Floyd's Theorem on the limitations of context free grammars and prove it using the Pumping Lemma.

[7 marks]

Question 2

(a) What is a data type? Explain in detail how a data type is modelled by a signature Σ and a class K of Σ-algebras A, and is defined by an axiomatic specification (Σ, T).

[5 marks]

(b) Define the concept of a *commutative ring*.

Which of the following properties are true for *all* commutative rings? Prove your answers.

(i) $(\forall x)[(x-5).(x-5) = x^2 - 10x + 25]$
(ii) $(\forall x, y)[x.y = 0 \Rightarrow x = 0 \text{ or } y = 0]$
(iii) $(\forall x, y)[(x+y).(x-y) = x^2 - y^2]$
(iv) $1 + 1 + 1 \neq 0$

[12 marks]

(c) Let A and B be Σ-algebras. Define the concept of a Σ-*isomorphism*

$$\varphi: A \to B.$$

How is the concept used in the theory of data types?

[4 marks]

(d) Let **N** be the set of natural numbers and let (**N**, 0, +) be an algebra of natural numbers. Which of the following functions φ: (**N**, 0, +) $\to$ (**N**, 0, +) is a homomorphism? Give reasons for your answers.

(i) $\varphi(x) = 5x$
(ii) $\varphi(x) = \min(x, 2)$
(iii) $\varphi(x) = 5$
(iv) $\varphi(x) = 0$

[4 marks]

Question 3

(a) Give a detailed construction of the semantics of the **while** programming language $WP(\sum)$ over any many sorted algebra A of signature $\sum$ using *input-output semantics*.

[10 marks]

(b) Illustrate the semantics in (a) by sketching how to define the input-output semantics of the Euclidean algorithm.

[5 marks]

(c) Discuss the general idea of defining a *kernel language* and its *extensions*. Illustrate this idea by adding the **case** statement to the language WP for **while** programs over all signatures.

[4 marks]

(d) Can one define the **if then else** construct from the other constructs of the **while** language? Give reasons for your answer. Is the **while** language "minimal" in the sense that if any one of its constructs is dropped then the language is weaker?

[6 marks]