CS_376 PROGRAMMING WITH ABSTRACT DATA TYPES

(Attempt 2 questions out of 3)

Question 1.

(a) (i) What does it mean for a closed formula to be *logically valid*? [2]

(ii) Let Σ be the signature with one sort, one constant, 0, and one binary operation, $+$. Let A be the Σ-algebra of natural numbers, $\{0, 1, 2, 3, \ldots\}$, where the constant 0 is interpreted by the number 0 and the operation $+$ is interpreted by addition.

For each of the following formulas decide whether it holds in the Σ-algebra A and whether it is logically valid:

(ii-1) $\forall x\, \exists y\, (x + y = 0)$ [2]

(ii-2) $\forall x\, (\forall y\, (x + y = y) \to \exists y\, (x + y = y))$ [2]

(ii-3) $\forall x\, (\exists y\, (x + y = y) \to \forall y\, (x + y = y))$ [2]

Justify your answer if you claim that a formula is not logically valid.

[8 marks]

(b) Use the predicates

$\mathsf{CS}(x) := x$ is a computer science student.

$\mathsf{WDA}(x) := x$ has won the WDA prize.

$\mathsf{PL}(x) := x$ does plagiarize.

to formalize the following statements:

(i) A computer science student has won the WDA prize. [3]

(ii) A computer science student does not plagiarize. [3]

[6 marks]

(c) (i) What is a *homomorphism* between two Σ-algebras? [2]

(ii) What is an *isomorphism* between two Σ-algebras? [1]

(iii) When are two Σ-algebras called *isomorphic*? [1]

(iv) What is an *equivalence* relation? [2]

(v) From the lectures we know the following facts:

(v-1) If $\varphi\colon A \to B$ is an isomorphism, then $\varphi^{-1}\colon B \to A$ is an isomorphism.

(v-2) If $\varphi_1\colon A \to B$ and $\varphi_2\colon B \to C$ are homomorphism, then so is $\varphi_2 \circ \varphi_1\colon A \to C$.

Use these facts to *prove* that the relation

$$A \simeq B \quad :\Leftrightarrow \quad A \text{ and } B \text{ are isomorphic}$$

is an *equivalence relation* on the class of all Σ-algebras. [5]

[11 marks]

Question 2.

(c) (i) What is an *abstract data type* (ADT)? [2]

(ii) When is an ADT called *monomorphic*? [2]

(iii) When is an ADT called *polymorphic*? [1]

[5 marks]

(b) (i) What does it mean that a Σ-algebra A is *initial* in a class $\mathcal{C}$ of Σ-algebras? [2]

(ii) What is an *initial specification*? [1]

(iii) What is a *model* of an initial specification? [2]

(iv) Sketch how a model of an inital specification can be constructed. [4]

(v) Is the ADT of all models of a given initial specification monomorphic or polymorphic? (no proof required). [1]

[10 marks]

(c) Give an initial specification of an editor which is able to perform the following actions:

(i) insert a character immediately to the left of the cursor; [1]

(ii) delete the character immediately to the left of the cursor (backspace); [1]

(iii) move the cursor one position to the right; [1]

(iv) move the cursor one position to the left; [1]

(v) move the cursor to beginning of file; [2]

(vi) move the cursor to end of file. [2]

You may (and should) import auxiliary data types (for example lists).

Clearly state which constants, operations and sorts are to be exported. [2]

[10 marks]

Question 3.

(a) Let Σ be a signature and X, Y sets of variables and $\theta\colon Y \to \mathrm{T}(\Sigma, X)$ a substitution.

For any term $t \in \mathrm{T}(\Sigma, Y)$ let

$t^{\mathrm{T}(\Sigma,X),\theta}$ denote the value of t in the term algebra $\mathrm{T}(\Sigma, X)$ under the assignment θ,

$t\theta$ denote the result of applying the substitution θ to t

(note the double role of θ).

Prove by induction on terms $t \in \mathrm{T}(\Sigma, Y)$ that

$$t^{\mathrm{T}(\Sigma,X),\theta} = t\theta$$

[7 marks]

(b) Derive the following formulas:

(i) $((P \to Q) \wedge \neg Q) \to \neg P$, in minimal logic. [3]

(ii) $(P \vee Q) \to (\neg P \to Q)$, in intuitionistic logic. [3]

(iii) $(\neg P \to \neg Q) \to (Q \to P)$, in classical logic. [3]

[9 marks]

(c) (i) What does it mean for a term rewriting system to be *terminating*? [2]

(ii) What does it mean for a term rewriting system to be *confluent*? [2]

(iii) Consider the signature $\Sigma := (\{s\}, \{c\colon s,\ f\colon s \to s\})$ and the term rewriting system R consisting of the only rule

$$f(f(x)) \mapsto c.$$

Is R is terminating? [2]

Is R is confluent? [3]

Justify your answers.

[9 marks]