Question 1

(a) Explain in detail how a data type is modelled by a signature Σ and a class K of Σ-algebras A, and is defined by an axiomatic specification (Σ, T).

[7 marks]

(b) Define the concept of a *commutative ring*. Give one infinite example of a commutative ring and and one finite example of a commutative ring.

Which of the following properties are true for *all* commutative rings? Give reasons for your answer.

(i) $(\forall x)[(x-2).(x-5) = x^2 - 7x + 10]$
(ii) $(\forall x, y)[x.y = 0 \text{ implies } x = 0 \text{ or } y = 0]$
(iii) $(\forall x, y)[(x-y).(x+y) = x^2 - y^2]$
(iv) $1 + 1 + 1 + 1 + 1 + 1 + 1 \neq 0$ (7 times)

[14 marks]

(c) Carefully describe an algebraic model of *one* of the following data types:
(i) data stores
(ii) streams representing data in time;
(iii) spatial objects representing data in space.

[4 marks]

Question 2

(a) Let A and B be Σ-algebras. Define carefully the concept of a Σ-*homomorphism*

$$\varphi: A \to B?$$

What is the concept used for in the theory of abstract data types?

[4 marks]

(b) Let **N** be the set of natural numbers and let (**N**, 0, +) be an algebra of natural numbers. Which of the following functions φ: (**N**, 0, +) $\to$ (**N**, 0, +) is a homomorphism? Give reasons for your answers.

(i) $\varphi(x) = 2x$
(ii) $\varphi(x) = 2(x+1)$
(iii) $\varphi(x) = \max(x, 2)$
(iv) $\varphi(x) = 2$
(v) $\varphi(x) = 0$

[5 marks]

(c) Consider the following signature Σ_{Proc} which models the idea of an interface to a data type of computational processes. There is one constant δ for a deadlocked process, and there are three operations that can be applied to processes p and q: *sequencing* p.q, *nondeterministic choice* p+q and *parallel execution* p||q.

signature Processes;
sorts proc;
constant δ: $\to$ proc
operations _._: proc $\times$ proc $\to$ proc;
_+_: proc $\times$ proc $\to$ proc;
_||_: proc $\times$ proc $\to$ proc;
endsig

Question continued …

We suppose any Σ_{Proc} algebra models a class of processes. Let P and R be two Σ_{Proc} algebras modelling two kinds of processes. Let

$$\varphi_P : P \to R$$

be a Σ_{Proc} homomorphism comparing P and R. Write down the *four* homomorphism equations for φ.

[8 marks]

(d) Define a *digital data type* using a homomorphism.

[8 marks]

Question 3

(a) What is a *formal language* L? Define the concept of a *context-free grammar G* and explain how it defines a formal language L(G). What is BNF notation?

[5 marks]

(b) Give a context-free grammar that defines the syntax of a language WP for defining *all* **while** programs over *all* signatures.

Sketch briefly the derivation of the following program from your grammar.

```
signature   clock;
sorts       time;
constant    0: → time;
operations  tick: time → time;
endsig

begin
            t := 0;
            while t ≥ 0 do t := tick(t) od
end
```

[8 marks]

(c) *Sketch briefly* how to define the semantics of the language WP. *Sketch briefly* the semantics of the clock program in (c).

[6 marks]

(d) Discuss the general idea of defining a *kernel language* and its *extensions*. Illustrate this idea by adding the **repeat** statement to the language WP for **while** programs over all signatures in (b).

[6 marks]

End