CS_342 2003/04
Constraint Satisfaction Problems and Applications

(Attempt 2 questions out of 3)

Question 1 Clause-sets, resolution and hitting clause-sets

(a) Basic notions

(i) Define variables ($\mathcal{VA}$), literals ($\mathcal{LIT}$), clauses ($\mathcal{CL}$) and clause-sets ($\mathcal{CLS}$), and explain the meaning of these concepts. **[4 marks]**

(ii) Define partial assignments ($\mathcal{PASS}$), and explain the operation $* : \mathcal{PASS} \times \mathcal{CLS} \to \mathcal{CLS}$ of partial assignments on clause-sets; that is, explain how $\varphi * F$ is computed from F. **[3 marks]**

(iii) Assume that, for some class $\mathcal{C} \subseteq \mathcal{CLS}$ of clause-sets, you have polynomial time SAT decidability. Under which circumstances is it possible to also find a satisfying assignment for $F \in \mathcal{C} \cap \mathcal{SAT}$ in polynomial time? How can this be done? **[4 marks]**

(b) Resolution

(i) Define the resolvent $C \diamond D$ of clauses C, D. **[2 marks]**

(ii) Show that

$$F = \{ \{\overline{a}, e\}, \{b, c, \overline{d}\}, \{d, \overline{f}\}, \{d, \overline{e}\}, \{b, \overline{c}\}, \{a, d, f\}, \{\overline{b}, \overline{d}\} \}.$$

is minimally unsatisfiable, where for showing unsatisfiability a resolution refutation must be exhibited. **[7 marks]**

(c) Hitting clause-sets

(i) Define the notion of a "hitting clause-set". **[1 marks]**

(ii) A *completion* of a hitting clause-set F is an unsatisfiable hitting clause-set F' with $F \subseteq F'$ and $\operatorname{var}(F) = \operatorname{var}(F')$. Prove that, for every hitting clause-set F, we can find a completion F' in polynomial time.

[4 marks]

Question 2 Formulating problems as satisfiability problems

(a) The Pigeonhole formulas

(i) Define the "Pigeonhole formulas" PHP_n^m and explain their meaning. **[4 marks]**

(ii) Decide for $m \geq 0$, $n \geq 1$ which clause-sets PHP_n^m are minimally unsatisfiable, and give reasons for your answer. **[5 marks]**

(b) Graph colouring

(i) Define when a graph is "k-colourable". **[1 marks]**

(ii) Describe a general encoding of the graph k-colourability problem as a satisfiability problem. **[5 marks]**

(c) Committee scheduling as SAT problems

Consider $p \in \mathbb{N}$ members of parliament, serving in $q \in \mathbb{N}$ committees as given by the $p \times q$ matrix S, where $S_{i,j} = 1$ if member i serves in committee j, while $S_{i,j} = 0$ otherwise. For some given set $\{1, \ldots, k\}$ of $k \in \mathbb{N}$ days, we are looking for an assignment of days to committees, such that every committee is assigned some day to meet, while if there is a common member of two committees, then these committees do not meet on the same day.

(i) Design propositional variables which can serve to encode the problem into propositional logic. **[2 marks]**

(ii) Using these variables, encode the problem as a clause-set. **[6 marks]**

(iii) Compute the standard measures n, c, ℓ and rk for your encoding. **[2 marks]**

Question 3

(a) Fast unit-clause propagation

(i) Define the "clause-literal graph" of a clause-set F. **[2 marks]**

(ii) Describe the linear-time procedure for computing $r_1(F)$ by means of the clause-literal graph of F. **[5 marks]**

(iii) Argue that this procedure actually runs in time linear in the input size. **[4 marks]**

(b) DLL-like SAT algorithms

(i) State the basic DLL-like SAT algorithm. **[4 marks]**

(ii) Discuss the places in this algorithm scheme where choices have to be performed, and give heuristical guidelines for good choices. **[5 marks]**

(c) Consider the class $\mathcal{C}$ of clause-sets $F \in \mathcal{CLS}$ such that, for each variable v, we have $\mathrm{vdg}_v(F) \le 2$. Show how we can decide satisfiability for $\mathcal{C}$ in polynomial time. **[5 marks]**