CS_232 2003/04
Algorithms and Complexity

(Attempt 2 questions out of 3)

Question 1 Graphs and reachability

(a) Fundamental notions

(i) Define "graphs" and "general graphs". **[2]**

(ii) Define "connected components" of a general graph. **[2]**

(iii) Define "trees". **[1]**

(iv) Define the notion of an "articulation point" in a connected general graph. **[1]**

[6 marks]

(b) Simple properties

(i) Characterise the articulation points of a tree (no proof required). **[2]**

(ii) An edge $e \in E(G)$ of a connected general graph G is called a "bridge" if removing e from G disconnects G (for example every edge in a path graph is a bridge, while no edge in a cycle graph is a bridge). Characterise bridges in terms of spanning trees, and prove that your characterisation is correct and complete. **[5]**

[7 marks]

(c) The `graph_traversal` procedure

(i) Describe precisely the `graph_traversal` procedure. **[6]**

(ii) Given a finite general graph G and a vertex $v \in V(G)$, describe how we can compute the table of distances $d_G(v, w) \in \mathbb{N}_0 \cup \{+\infty\}$ for $w \in V(G)$ in linear time. **[3]**

(iii) Given a finite connected graph G together with an edge labelling $w : E(G) \to \mathbb{R}$, describe how a spanning tree of G with minimal weight can be computed in quasi-linear time. **[3]**

[12 marks]

Question 2 Isomorphic graphs and vertex colouring

(a) Graph isomorphism

(i) Define when two general graphs G_1, G_2 are called "isomorphic". **[2]**

(ii) Describe in general terms a (simple) procedure for deciding whether two graphs are isomorphic or not, and give reasonable upper bounds on its time and space complexity (ignoring some small polynomial factor in the size of the input). **[3]**

(iii) Explain how vertex degrees (sometimes) can be used to distinguish between non-isomorphic graphs **[2]**

(iv) Consider a graph G with six vertices and seven edges, which is obtained from C^6 by adding one edge. Determine the isomorphism types of G, i.e., give a finite list of possible G's and show that any two graphs from this list are non-isomorphic and that the list is complete. **[3]**

[10 marks]

(b) Graph colouring

(i) Define the notion of a "vertex colouring" for a graph G, and define the chromatic number $\chi(G)$. **[2]**

(ii) Describe in general terms a (simple) procedure for computing $\chi(G)$ for finite graphs G, and give reasonable upper bounds on its time complexity depending (only) on the number $n := |V(G)|$ of vertices (ignoring some small polynomial factor in the size of the input). **[4]**

(iii) Describe the "traversing" greedy graph colouring algorithm and state the argument proving that the colouring computed is optimal for input graphs G with $\chi(G) \leq 2$ (you may assume that G is connected). **[4]**

(iv) An "edge-colouring" of a graph G is a map $f : E(G) \to C$ from the set of edges of G into some set C of "colours" such that adjacent edges get different colours. How can the edge-colouring problem be reduced to a vertex-colouring problem? **[1]**

(v) Determine $\chi(T)$ for all trees T. **[2]**

(vi) Consider the "chessboard graph" for an $n \times n$ chessboard, where the nodes are the squares, and two squares are connected by an edge if they are horizontal or vertical neighbours. Draw the graph for $n = 4$. Are these graphs bipartite? (Give reasons for your answer.) **[2]**

[15 marks]

Question 3 Problems in NP

(a) Complexity classes

(i) Define the complexity classes P, NP and co-NP. **[4]**

(ii) Define NP-completeness. **[2]**

(iii) State for the following problems whether they are in P, in NP or whether they are NP-complete (choose in each case the most precise answer):

1. Is a graph connected?
2. Are two graphs isomorphic?
3. Is a propositional formula in conjunctive normal form satisfiable?
4. Is a graph 2-colourable?
5. Is a graph 3-colourable?

[3]

[9 marks]

(b) The SAT problem

(i) Define the SAT problem. **[3]**

(ii) Decide for each of the following clause-sets F_i, $i = 1, 2$, whether F_i is satisfiable or not, and justify your answers:

$$\begin{aligned} F_1 &:= \{\, \{\overline{a}, \overline{b}, \overline{c}\}, \{\overline{a}, \overline{b}, c\}, \{a\}, \{\overline{a}, b\} \,\} \\ F_2 &:= \{\, \{a, b\}, \{\overline{a}, b, c\}, \{\overline{b}, c\}, \{\overline{a}, \overline{c}\} \,\}. \end{aligned}$$

[4]

(iii) Describe how the k-colouring problem can be reduced in polynomial time to the k-SAT problem. **[4]**

(iv) Describe the "clause-literal graph", and how we can use the clause-literal graph to perform unit clause propagation in linear time. **[5]**

[16 marks]