CS_226
COMPUTABILITY THEORY
Exam May/June 2006

(*Attempt* **2** *questions out of 3*)

Question 1.

(a) What does it mean for two sets to be equinumerous (i.e. to have the same cardinality)? Is it possible for a finite set to be equinumerous with $\mathbb{N}$? Justify your answer. Give an example of an infinite set which is not equinumerous with $\mathbb{N}$. Justify your answer.

[5 marks]

(b) Show directly (without referring to lemmas and theorems shown in the lecture) that $\mathcal{P}(\mathbb{N})$ is uncountable.

Hint: Show first directly that $\mathbb{N} \not\approx \mathcal{P}(\mathbb{N})$ ($\mathbb{N}$ is not equinumerous with $\mathcal{P}(\mathbb{N})$). What else does one need to show?

[5 marks]

(c) Which of the following sets are countable? Justify your answers.

(i) The power set of $\{0, 1, 2, 3, 4\}$.
(ii) The power set of $\mathbb{Z}$.
(iii) The set of finite subsets of $\mathbb{Z}$.
(iv) The set of infinite subsets of $\mathbb{Z}$.

[8 marks]

(d) Let $h : \mathbb{N}^2 \to \{0, 1\}$ be a total function such that, for every computable function $f : \mathbb{N} \to \{0, 1\}$, there exists a number $e \in \mathbb{N}$ such that $h(e, n) = f(n)$ for all n. Show that h is non-computable. Please note that both f and h have results in $\{0, 1\}$.

[7 marks]

Please turn over for Question 2.

Question 2.

(a) Define the notion of a Turing machine. Your definition should include an explanation of the meaning of a Turing machine instruction. Define as well the k-ary function $\mathrm{T}^{(k)} : \mathbb{N}^k \stackrel{\sim}{\to} \mathbb{N}$ computed by a Turing machine T.

[6 marks]

(b) Consider a Turing machine with alphabet $\Sigma := \{0, 1, \sqcup\}$, where $\sqcup$ is the symbol for blank, states $\{s_0, s_1, s_2, s_3, s_4\}$, initial state s_0, and with the following instructions:

- $(s_0, 0, s_0, 0, \mathsf{R})$,
- $(s_0, 1, s_0, 1, \mathsf{R})$,
- $(s_0, \sqcup, s_1, \sqcup, \mathsf{L})$,
- $(s_1, 1, s_1, 0, \mathsf{L})$,
- $(s_1, 0, s_2, 1, \mathsf{L})$,
- $(s_2, 0, s_2, 0, \mathsf{L})$,
- $(s_2, 1, s_2, 1, \mathsf{L})$,
- $(s_2, \sqcup, s_4, \sqcup, \mathsf{R})$,
- $(s_1, \sqcup, s_3, 1, \mathsf{L})$,
- $(s_3, \sqcup, s_4, \sqcup, \mathsf{R})$.

By digits we mean in the following the symbols 0 and 1.

(i) Describe the steps carried out by a Turing machine, if it is run with the tape initially containing a non-empty string of digits, with blanks to the left and right of the string, and the head initially positioned at the left most digit.

(ii) Determine the function $f : \mathbb{N} \stackrel{\sim}{\to} \mathbb{N}$ computed by this Turing machine. Justify your answer.

[9 marks]

(c) Introduce a URM program using instructions `succ(k)`, `pred(k)`, `ifzero(j,k)` only, which computes the function $f : \mathbb{N}^2 \to \mathbb{N}$, $f(x, y) := x \dot{-} y$. Explain why your program computes this function.

[6 marks]

(d) In the lectures, a direct proof was given that every URM computable function is Turing Machine computable. This was done by simulating URMs by Turing Machines. The opposite direction, namely that every Turing Machine computable function is URM computable, was shown by an indirect argument. Outline this argument. You do not need to carry out any technical details.

[4 marks]

Question 3 follows on the next page.

Question 3.

(a) (i) State Kleene's recursion theorem.

(ii) Use Kleene's recursion theorem in order to show that the function $\mathrm{fib} : \mathbb{N} \to \mathbb{N}$ enumerating the Fibonacci numbers is computable. Here $\mathrm{fib}(0) := \mathrm{fib}(1) := 1$, $\mathrm{fib}(n+2) := \mathrm{fib}(n) + \mathrm{fib}(n+1)$.

[6 marks]

(b) (i) What is a recursively enumerable set?

(ii) Show that the intersection of two recursively enumerable sets is recursively enumerable.

[6 marks]

(c) Let P, Q be primitive-recursive predicates on $\mathbb{N}$, and $f, g, h : \mathbb{N} \to \mathbb{N}$ be primitive recursive. Show that $k : \mathbb{N} \to \mathbb{N}$,

$$k(n) := \begin{cases} f(n) & \text{if } P(n), \\ g(n) & \text{if } \neg P(n) \wedge Q(n), \\ h(n) & \text{if } \neg P(n) \wedge \neg Q(n) \end{cases}$$

is primitive recursive.

[6 marks]

(d) Show that the predicate Prime on the natural numbers, given as

$$\mathrm{Prime}(n) :\Leftrightarrow n \text{ is a prime number}$$

is primitive-recursive.

Hint: Express "Prime(n)" as a formula using bounded quantifiers only.

[7 marks]