CS_226
COMPUTABILITY THEORY

(*Attempt* **2** *questions out of 3*)

Question 1.

(a) What does it mean for two sets to have the same cardinality? Explain the relationship of this notion with the property of two finite sets having the same number of elements.

[4 marks]

(b) Give a proof that $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ do not have the same cardinality.

[5 marks]

(c) Which of the following sets are countable? Justify your answers.

- The set of irrational numbers.
- The set of natural numbers greater than 3.
- The set of infinite subsets of $\mathbb{N}$.

[9 marks]

(d) Let $h : \mathbb{N}^2 \to \mathbb{N}$ be a total function such that, for every computable function f, there exists a number $e \in \mathbb{N}$ such that $h(e, n) = f(n)$ for all n. Show that h is non-computable.

[7 marks]

Question 2.

(a) Define the notion of a Turing machine. Your definition should include an explanation of the meaning of a Turing machine instruction.

[6 marks]

(b) Introduce a Turing machine over the alphabet $\{0, 1, \sqcup\}$, which operates as follows: Assume that initially the tape contains a binary string (i.e. an element $\{0, 1\}^*$), all other cells are blank, and the head points to any of the digits of the binary string. When started with this configuration, the Turing machine should eventually terminate, with the tape containing the result of interchanging zeros and ones in the initial string, all other cells being blank, and the head pointing to the leftmost digit. For instance, if the tape originally contained the binary string 001100, then at the end it should contain the binary string 110011. Explain, how you obtained your solution.

[6 marks]

(c) (i) Assume P is a URM program. Define how to compute the corresponding partial function $P^{(k)} : \mathbb{N}^k \stackrel{\sim}{\rightarrow} \mathbb{N}$.

(ii) Consider the following URM program $P = \mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2$:

$\mathrm{I}_0 = \texttt{ifzero}(0, 3)$
$\mathrm{I}_1 = \texttt{succ}(0)$
$\mathrm{I}_2 = \texttt{ifzero}(1, 0)$.

Describe the function $P^{(2)}$ computed by P, and justify your answer.

[8 marks]

(d) Assume $f : \mathbb{N}^2 \stackrel{\sim}{\rightarrow} \mathbb{N}$ is URM-computable. Describe, how one can obtain a program computing $g : \mathbb{N} \stackrel{\sim}{\rightarrow} \mathbb{N}$, $g(x) :\simeq (\mu y.f(x, y) \simeq 0)$. You can make use of higher programming constructs for URMs, as introduced in the lecture. Explain your result.

[5 marks]

Question 3.

(a) What is a primitive-recursive function? Are all primitive-recursive functions computable? Are all computable functions primitive-recursive?

[8 marks]

(b) Show, using the definition of primitive-recursive functions directly, that the function

$$f : \mathbb{N} \to \mathbb{N} \ , \quad f(x) := 3x$$

is primitive-recursive.

[4 marks]

(c) Let P be a primitive-recursive predicate on $\mathbb{N}$. Show that the function $f : \mathbb{N} \to \mathbb{N}$, defined as

$$f(n) := \begin{cases} \max\{k \leq n \mid P(k)\}, & \text{if there exists } k \leq n \text{ such that } P(k) \text{ holds,} \\ n+1, & \text{otherwise,} \end{cases}$$

is primitive-recursive.

[5 marks]

(d) (i) What is a recursive subset of $\mathbb{N}^n$?

(ii) Which of the following statements are true:

* If $A \subseteq \mathbb{N}$ is recursive, so is $\mathbb{N} \setminus A$.
* If $A, B \subseteq \mathbb{N}$ are recursive, so is $A \cup B$.
* If $A \subseteq \mathbb{N}^2$ is recursive, so is $\{n \in \mathbb{N} \mid \exists m.(n, m) \in A\}$.

Justify your answers.

[8 marks]